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Stability of a non-commutative Jackiw–Teitelboim gravity

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Abstract. We start with a non-commutative version of the Jackiw–Teitelboim gravity in two dimensions which has a linear potential for the dilaton fields. We study whether it is possible to deform this model by adding quadratic terms to the potential but preserving the number of gauge symmetries. We find that no such deformation exists (provided one does not twist the gauge symmetries).

1 Introduction

Dilaton gravities in two dimensions [1] are a good testing ground for many theoretical ideas also relevant in higher dimensions. After some field redefinitions almost all interesting models of that type can be written in the form

$$S = \int \mathrm{d}^2 x \varepsilon^{\mu\nu} \left(\phi \partial_\mu \omega_\nu + \phi_a D_\mu e^a_\nu - \varepsilon_{ab} e^a_\mu e^a_\nu V(\phi) \right) , \quad (1)$$

where e^a_μ is the zweibein, $\varepsilon^{\mu\nu}$ is the Levi–Civita symbol (see Appendix A for our sign conventions). The covariant derivative

$$\varepsilon^{\mu\nu} D_\mu e^a_\nu = \varepsilon^{\mu\nu} \left(\partial_\mu e^a_\nu + \omega_\mu \varepsilon^a{}_b e^b_\nu \right) \quad (2)$$

contains the spin connection $\omega_\mu \varepsilon^a{}_b$. Here ϕ is a scalar field called the dilaton. ϕ_a is an auxiliary field. In the commutative case, which we are considering at the moment, any choice of the potential $V(\phi)$ leads to a consistent model. Two examples are of particular importance for us. A constant potential V corresponds to the (conformally transformed) string gravity, also called the Witten black hole [2]. For a linear potential, $V(\phi) \propto \phi$, one gets the Jackiw–Teitelboim (JT) model [3], whose equations of motion were studied earlier in [5].

The auxiliary field ϕ_a generates the condition that ω_μ is the Levi–Civita connection compatible with e^a_μ. Under this condition $\varepsilon^{\mu\nu}\partial_\mu\omega_\nu$ becomes proportional to the usual Riemann curvature (the terms proportional to ϕ_a, of course, disappear). In this way one arrives at a second order formalism, which may be more familiar to some of the readers. However, the first order action (1) has many advantages over the second order one. For instance, the classical equations of motion are much easier to solve [6], and in the quantum case, it is possible to perform the path integral over the geometric variables even in the presence of additional matter fields [7].

In this paper we study which models of 2D dilaton gravity can be formulated on non-commutative spaces. Let us define the star product of functions which will replace the usual point-wise multiplication. The Moyal star product of functions on $\mathbb{R}^2$ reads

$$f \star g = f(x) \exp\left(\frac{\mathrm{i}}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x) . \quad (3)$$

θ is a constant antisymmetric matrix. This product is associative, $(f \star g) \star h = f \star (g \star h)$. In this form the star product has to be applied to plane waves and then extended to all (square integrable) functions by means of the Fourier series [8]. Obviously,

$$x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu} . \quad (4)$$

Furthermore, the Moyal product is closed,

$$\int_{\mathcal{M}} \mathrm{d}^2 x f \star g = \int_{\mathcal{M}} \mathrm{d}^2 x f \times g \quad (5)$$

(where $\times$ denotes the usual point-wise product), it respects the Leibniz rule

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g + f \star (\partial_\mu g) \quad (6)$$

and allows one to make cyclic permutations under an integral:

$$\int_{\mathcal{M}} \mathrm{d}^2 x f \star g \star h = \int_{\mathcal{M}} \mathrm{d}^2 x h \star f \star g . \quad (7)$$

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The complex conjugation reverses the order of factors,

$$(f \star g)^* = g^* \star f^* \,. \tag{8}$$

The product (3) is not the only possible choice of an associative non-commutative product. The right hand side of (4) can depend, in principle, on the coordinates.

An important step towards constructing a satisfactory non-commutative gravity was recently made by Wess and collaborators [9], who understood how one can construct diffeomorphism invariants, including the Einstein–Hilbert action, on non-commutative spaces (see also [10] for a real formulation). There is, however, a price to pay. The diffeomorphism group becomes twisted; there is a non-trivial coproduct due to which the action of the symmetries on tensor products looks very unusual [11, 12].

In two dimensions it is possible to construct non-commutative (dilaton) gravity models with an usual (non-twisted) realization of gauge symmetries. A non-commutative version of the Jackiw–Teitelboim (NCJT) model was constructed in [13] and then quantized in [14]. A non-commutative Witten black hole model was suggested in [15]. Both these models are of the Yang–Mills type: the JT model is equivalent to a topological BF model; the Witten black hole may be represented as a Wess–Zumino–Novikov–Witten model. There are some general procedures of how such models can be formulated in the non-commutative case [13, 16]. It is important therefore to check whether one can go beyond the Yang–Mills paradigm. Besides, if we are on the right track, dilaton gravities should exist not only for linear or constant potentials, but also for an arbitrary potential V. In the present paper we study whether quadratic potentials are allowed.

To analyze the gauge symmetries we use the canonical formalism for non-commutative spacetime developed in [15]. This is not a canonical formalism in the usual sense of the word[1] [17, 18], but it makes it possible to define the notion of first-class constraints and to associate a gauge symmetry to them. As to commutative gauge theories, it was conjectured by Dirac that all first-class constraints act as generators of gauge transformations. For some classes of commutative gauge theories this conjecture can be proved and, in addition, it turns out that the number of independent non-trivial gauge transformations is equal to the number of primary first-class constraints [17]. The symmetry structure of a general commutative gauge theory was recently described in detail and related to the constraint structure of the theory in the Hamiltonian formulation [19]. In particular, the gauge charge was constructed explicitly as a decomposition in the special orthogonal constraint basis. It was demonstrated that, in the general case, the gauge charge cannot be constructed with the help of first-class constraints alone, for its decomposition also contains special combinations of second-class constraints.

[1] Since the space-time non-commutative theories are non-local in time and contain an infinite number of time derivatives hidden in the star product, it is obvious that some modification of the standard canonical formalism is necessary.

Consider those classical actions which can be represented in the form

$$S = \int \mathrm{d}^2x \left(p^i \partial_0 q_i - \lambda^i \star G_i(p,q) \right) \,, \tag{9}$$

so that the expressions ("constraints") $G_i(p,q)$ do not contain explicit time derivatives (implicit time derivatives are always present through the star product). The paper [15] demonstrated that one can define the canonical pairs ignoring implicit time derivatives in the star product. In this sense the p^i become canonically conjugated to q_i. The brackets are then defined by the equation

$$\{q_i(x), p^j(y)\} = \delta_i^j \delta^2(x-y) \,. \tag{10}$$

This definition can be extended to all polynomial functionals on the phase space [15]. If the brackets between the constraints are again linear combinations of constraints, then the non-commutative action has a gauge symmetry associated to each G_i. In this sense, the G_i may be called first-class constraints.

The most unusual property of the bracket (10) is the presence of the delta-function of the time coordinates on the right hand side. However, since the space-time non-commutative theories are non-local in the time direction, restriction of the brackets of the phase space variables calculated at the same value of time does not look natural and even consistent. The presence of an additional delta-function in (10) reminds us of the Ostrogradski formalism for the theories with higher temporal derivatives (see [20, 21] and [15] for a more extensive discussion). Anyway, one can also use the brackets (10) to analyze gauge symmetries in commutative theories. It is not clear, however, whether one can use the modified brackets for quantization. In the present paper we shall exclusively use (10) to define the Poisson structure.

We shall demonstrate that one cannot consistently add quadratic terms to the dilaton potential of the NCJT model, so that it is stable against such deformations.

2 Non-commutative Jackiw–Teitelboim gravity

A non-commutative version of the Jackiw–Teitelboim model has been constructed in [13]. It has been identified with a $U(1,1)$ gauge theory on non-commutative $\mathbb{R}^2$. The action reads

$$S^{(0)} = \frac{1}{4} \int \mathrm{d}^2x \, \varepsilon^{\mu\nu} \left[\phi_{ab} \star \left(R^{ab}_{\mu\nu} - 2\Lambda e^a_\mu \star e^b_\nu \right) - 2\phi_a \star T^a_{\mu\nu} \right] \,, \tag{11}$$

with curvature tensor

$$\begin{aligned} R^{ab}_{\mu\nu} =& \varepsilon^{ab} \left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \frac{\mathrm{i}}{2}[\omega_\mu, b_\nu] + \frac{\mathrm{i}}{2}[b_\mu, \omega_\nu] \right) \\ &+ \eta^{ab} \left(\mathrm{i}\partial_\mu b_\nu - \mathrm{i}\partial_\nu b_\mu + \frac{1}{2}[\omega_\mu, \omega_\nu] - \frac{1}{2}[b_\mu, b_\nu] \right) \,, \end{aligned} \tag{12}$$

and with non-commutative torsion

$$T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \frac{1}{2}\varepsilon^a{}_b \left([\omega_\mu, e^b_\nu]_+ - [\omega_\nu, e^b_\mu]_+\right) + \frac{\mathrm{i}}{2}\left([b_\mu, e^a_\nu] - [b_\nu, e^a_\mu]\right) . \tag{13}$$

There are two dilaton fields, ϕ and ψ, which are combined into

$$\phi_{ab} := \phi\varepsilon_{ab} - \mathrm{i}\eta_{ab}\psi . \tag{14}$$

All commutators (denoted by square brackets) and anti-commutators (denoted by $[\ ,\ ]_+$) are calculated with the Moyal star product.

We note that (11) contains more fields than the initial commutative model. This is related to the fact that the gauge group of the commutative JT model, which is $SU(1,1)$, cannot be closed on the non-commutative plane. To make the closure, one introduces additional $U(1)$ fields ψ and b_μ which decouple in the commutative limit.

One can rewrite (11) in the canonical form:

$$S^{(0)} = \int \mathrm{d}^2x \left(p^i\partial_0 q_i - \lambda^i \star G^{(0)}_i\right) , \tag{15}$$

where

$$\begin{aligned} q_i &= (e^a_1, \omega_1, b_1) , \\ p^i &= (\phi_a, \phi, -\psi) , , \\ \lambda^i &= (e^a_0, \omega_0, b_0) . \end{aligned} \tag{16}$$

The constraints are

$$G^{(0)}_a = -\partial_1\phi_a + \frac{1}{2}\varepsilon^b{}_a[\omega_1, \phi_b]_+ + \frac{\mathrm{i}}{2}[\phi_a, b_1] + \frac{\Lambda}{2}\left(-\varepsilon_{ab}[e^b_1, \phi]_+ + \mathrm{i}\eta_{ab}[e^b_1, \psi]\right) , \tag{17}$$

$$G^{(0)}_3 = -\partial_1\phi + \frac{\mathrm{i}}{2}[\phi, b_1] + \frac{\mathrm{i}}{2}[\psi, \omega_1] - \frac{1}{2}\varepsilon^a{}_b[\phi_a, e^b_1]_+ , \tag{18}$$

$$G^{(0)}_4 = \partial_1\psi - \frac{\mathrm{i}}{2}[\psi, b_1] + \frac{\mathrm{i}}{2}[\phi, \omega_1] + \frac{\mathrm{i}}{2}[\phi_a, e^a_1] . \tag{19}$$

One can check that the constraint algebra closes, and the brackets between the constraints read

$$\left\{\int \alpha^a \star G^{(0)}_a, \int \beta^b \star G^{(0)}_b\right\} = -\frac{\Lambda}{2}\int\left(\varepsilon_{ab}[\alpha^a, \beta^b]_+ \star G^{(0)}_3 + \mathrm{i}[\alpha_a, \beta^a] \star G^{(0)}_4\right) , \tag{20}$$

$$\left\{\int \alpha \star G^{(0)}_3, \int \beta \star G^{(0)}_3\right\} = \frac{\mathrm{i}}{2}\int [\alpha, \beta] \star G^{(0)}_4 , \tag{21}$$

$$\left\{\int \alpha \star G^{(0)}_4, \int \beta \star G^{(0)}_4\right\} = -\frac{\mathrm{i}}{2}\int [\alpha, \beta] \star G^{(0)}_4 , \tag{22}$$

$$\left\{\int \alpha \star G^{(0)}_3, \int \beta \star G^{(0)}_4\right\} = -\frac{\mathrm{i}}{2}\int [\alpha, \beta] \star G^{(0)}_3 \tag{23}$$

$$\left\{\int \alpha \star G^{(0)}_3, \int \beta^a \star G^{(0)}_a\right\} = -\frac{1}{2}\int [\alpha, \beta^a]_+ \, \varepsilon^b{}_a \star G^{(0)}_b , \tag{24}$$

$$\left\{\int \alpha \star G^{(0)}_4, \int \beta^a \star G^{(0)}_a\right\} = -\frac{\mathrm{i}}{2}\int [\alpha, \beta^a] \star G^{(0)}_a . \tag{25}$$

Here we introduced a short-hand notation $\int := \int \mathrm{d}^2x$.

3 Deformations

Let us now discuss deformations of the NCJT model. We shall add some terms to the action (11) so that (i) the field content of the model will not be changed, and (ii) the number of secondary first class constraints (and, consequently, the number of gauge symmetries) will also remain invariant. Being inspired by commutative dilaton gravity models we only consider the deformations of the potential term, and we only add terms of the next (quadratic) order in the two dilaton fields ϕ and ψ.

In addition to analogies with the commutative case, there are also other reasons for not considering deformations of the curvature and torsion terms. For example, replacing ϕ_{ab} in (14) by a non-linear function of the dilatons is equivalent to a redefinition of the dilaton fields. Adding higher powers of the curvature in general adds new degrees of freedom to the theory, and this is a more drastic modification than the usual understanding as deformations. The same also holds for torsion terms.

Further restrictions on possible deformations are imposed by the global symmetries of the model which we would like to preserve. First of all, we require the symmetry with respect to global rotation of the tangential and world indices. This implies that all indices must be contracted pair-wise. We also require that the terms being added are of even parity. Since ϕ is a scalar, and ψ is a pseudo-scalar, even (odd) powers of ψ should be multiplied with even (odd) powers of the Levi–Civita symbol ε. As a result, we obtain the following family of quadratic deformations of the NCJT model:

$$S = S^{(0)} + \tilde{S} , \tag{26}$$

where

$$\begin{aligned}\tilde{S} = \int \mathrm{d}^2x \Big(&\varepsilon^{\mu\nu}\varepsilon_{ab}\big(c_1 e^a_\mu \star e^b_\nu \star \phi^2 + c_2 e^a_\mu \star e^b_\nu \star \psi^2 \\ &+ c_3 e^a_\mu \star \phi \star e^b_\nu \star \phi + c_4 e^a_\mu \star \psi \star e^b_\nu \star \psi\big) \\ &+ \varepsilon^{\mu\nu}\eta_{ab}\Big(c_5 e^a_\mu \star e^b_\nu \star [\phi, \psi] + \mathrm{i}c_6 e^a_\mu \star e^b_\nu \star [\phi, \psi]_+ \\ &+ \frac{\mathrm{i}}{2}c_7(e^a_\mu \star \phi \star e^b_\nu \star \psi - e^a_\mu \star \psi \star e^b_\nu \star \phi)\Big)\Big) .\end{aligned} \tag{27}$$

The arbitrary constants c_1, c_2, ..., c_7 must be real to preserve the reality of the total action S. The powers are taken with the star product, for example $\phi^2 \equiv \phi \star \phi$.

The constraints read

$$G_a = G_a^{(0)} + \tilde{G}_a, \quad G_3 = G_3^{(0)}, \quad G_4 = G_4^{(0)}, \tag{28}$$

where

$$\tilde{G}_a = \varepsilon_{ab}\left(c_1[e_1^b,\phi^2]_+ + c_2[e_1^b,\psi^2]_+ + 2c_3\phi e_1^b\phi + 2c_4\psi e_1^b\psi\right) + \eta_{ab}\left(c_5[e_1^b,[\phi,\psi]] + \mathrm{i}c_6[e_1^b,[\phi,\psi]_+] + \mathrm{i}c_7(\phi e_1^b\psi - \psi e_1^b\phi)\right). \tag{29}$$

Our next step is to check whether the constraint algebra still closes on the constraint surface [2]. Since the constraints G_3 and G_4 are unchanged, the brackets between them, (21)–(23), are the same. It is an easy exercise to check that for all values of the constants c_m

$$\left\{\int \alpha\star G_4, \int \beta^a\star\tilde{G}_a\right\} = -\frac{\mathrm{i}}{2}\int[\alpha,\beta^a]\star\tilde{G}_a. \tag{30}$$

Consequently, for any values of c_m the bracket between $G_4$and G_a,

$$\left\{\int \alpha\star G_4, \int \beta^a\star G_a\right\} = -\frac{\mathrm{i}}{2}\int[\alpha,\beta^a]\star G_a, \tag{31}$$

is again a constraint in the new set (28), so that we are getting no restrictions on c_m.

Let us now consider the bracket between G_3 and G_a,

$$\begin{aligned}\left\{\int \alpha\star G_3, \int \beta^a\star\tilde{G}_a\right\} = \frac{1}{2}\int\Big[&c_1\left(\beta_a\star[[\alpha,e_1^a]_+,\phi^2]_+ + \mathrm{i}\beta^a\star\varepsilon_{ab}[e_1^b,[[\alpha,\psi],\phi]_+]_+\right) + c_2\left(\beta_a\star[[\alpha,e_1^a]_+,\psi^2]_+ - \mathrm{i}\beta^a\star\varepsilon_{ab}[e_1^b,[[\alpha,\phi],\psi]_+]_+\right) \\ &+ 2c_3\left(\beta_a\star\phi\star[\alpha,e_1^a]_+\star\phi + \mathrm{i}\beta^a\star\varepsilon_{ab}([\alpha,\psi]\star e_1^b\star\phi + \phi\star e_1^b\star[\alpha,\psi])\right) \\ &+ 2c_4\left(\beta_a\star\psi\star[\alpha,e_1^a]_+\star\psi - \mathrm{i}\beta^a\star\varepsilon_{ab}([\alpha,\phi]\star e_1^b\star\psi + \psi\star e_1^b\star[\alpha,\phi])\right) \\ &+ c_5\left(\beta^a\star\varepsilon_{ab}[[\alpha,e_1^b]_+,[\phi,\psi]] + \mathrm{i}\beta_a\star[e_1^a,[[\alpha,\psi],\psi]-[\phi,[\alpha,\phi]]]\right) \\ &+ \mathrm{i}c_6\left(\beta^a\star\varepsilon_{ab}[[\alpha,e_1^b]_+,[\phi,\psi]_+] + \mathrm{i}\beta_a\star[e_1^a,[[\alpha,\psi],\psi]_+ - [\phi,[\alpha,\phi]]_+]\right) \\ &+ \mathrm{i}c_7\left(\beta^a\star\varepsilon_{ab}(\phi\star[\alpha,e_1^b]_+\star\psi - \psi\star[\alpha,e_1^b]_+\star\phi) + \mathrm{i}\beta_a\star([\alpha,\psi]\star e_1^a\star\psi - \phi\star e_1^a\star[\alpha,\phi] + [\alpha,\phi]\star e_1^a\star\phi - \psi\star e_1^a\star[\alpha,\psi])\right)\Big]. \end{aligned} \tag{32}$$

First we observe that the right hand side of (32) contains no terms with derivatives. This excludes the possibility of the bracket (32) containing any terms proportional to (17), (18), or (19). Therefore, this bracket can only be proportional to (29), with coefficients (structure functions) as in (24), so that the bracket between G_3 and G_a sums up to become

$$\left\{\int \alpha\star G_3, \int \beta^a\star G_a\right\} = -\frac{1}{2}\int[\alpha,\beta^a]_+\,\varepsilon^b{}_a\star G_b. \tag{33}$$

We have to compare the expressions on both sides of (33) to get restrictions on the constants c_m. There are no monomials on the right hand side of (33) which are second order in ϕ and have an explicit i factor. At the same time, there is such a term proportional to c_5 in (32). Since all c_m are real, we conclude that

$$c_5 = 0. \tag{34}$$

Next we compare the terms in which two ϕ appear next to each other [3] (combined in ϕ^2). Those terms agree on both sides of (33) if and only if

$$c_6 = -c_1. \tag{35}$$

By comparing the terms where two fields ϕ appear separated by other fields, we obtain the following condition :

$$2c_3 = -c_7. \tag{36}$$

Then we repeat the same procedure with the terms which are quadratic in ψ to get

$$c_2 = c_6, \qquad 2c_4 = -c_7. \tag{37}$$

The comparison of mixed terms (containing both ϕ and ψ) does not produce any additional restrictions on c_m. We conclude that only two independent constants (say, c_1 and c_7) remain, so $\tilde{G}_a$ can be rewritten as

$$\begin{aligned}\tilde{G}_a = &c_1\left(\varepsilon_{ab}[e_1^b,\phi^2-\psi^2]_+ - \mathrm{i}\eta_{ab}[e_1^b,[\phi,\psi]_+]\right) \\ &+ c_7\left(-\varepsilon_{ab}(\phi\star e_1^b\star\phi + \psi\star e_1^b\star\psi) + \mathrm{i}\eta_{ab}(\phi\star e_1^b\star\psi - \psi\star e_1^b\star\phi)\right). \end{aligned} \tag{38}$$

It remains to study the brackets between G_a and G_b. Obviously, the brackets between $\tilde{G}_a$ and $\tilde{G}_b$ vanish, so that all new information is contained in the brackets between $G_a^{(0)}$ and $\tilde{G}_b$. The strategy is the same as above. First we analyze the derivative terms:

$$\begin{aligned}&\left\{\int \alpha^a\star G_a^{(0)}, \int \beta^b\star\tilde{G}_b\right\} + \left\{\int \alpha^a\star\tilde{G}_a, \int \beta^b\star G_b^{(0)}\right\} \\ &= \int\Big[c_1\left(\partial_1\phi\star([\phi,\varepsilon_{bc}[\beta^b,\alpha^c]_+]_+ + \mathrm{i}[\psi,[\alpha_b,\beta^b]]_+) + \partial_1\psi\star(-[\psi,\varepsilon_{bc}[\beta^b,\alpha^c]_+]_+ + \mathrm{i}[\phi,[\alpha_b,\beta^b]]_+)\right) \\ &\quad + c_7\left(\partial_1\phi\star(-\varepsilon_{bc}(\beta^b\star\phi\star\alpha^c + \alpha^c\star\phi\star\beta^b) + \mathrm{i}(\alpha^b\star\psi\star\beta_b - \beta_b\star\psi\star\alpha^b)) - \partial_1\psi\star(\varepsilon_{bc}(\beta^b\star\psi\star\alpha^c + \alpha^c\star\psi\star\beta^b) + \mathrm{i}(\alpha^b\star\phi\star\beta_b - \beta_b\star\phi\star\alpha^b))\right)\Big] \\ &\quad + \text{non-derivative terms}. \end{aligned} \tag{39}$$

[2] In principle, other substantial modifications of the constraint algebra may occur, but not in the present case. We limit the number of gauge symmetries to four, so only four first-class constraints are allowed, because there are only four canonical pairs of variables. Therefore, the only possibility is that G_i are first class and that their brackets give again linear combinations of G_i.

[3] This also includes the terms which can be put in this form by using property (7).

From this equation we see that, since the bracket between G_a and G_b must be a linear combination of the constraints (28), the constraints appearing on the right hand side can only be G_3 and G_4, since the derivative $\partial_1\phi_a$ belonging to G_a is not present. In fact, one can also obtain the structure functions from (39), but their precise form will not be needed. Let us consider the terms in the bracket which contain the zweibein e_1^a and the dilaton ϕ:

$$\begin{aligned}&\left\{\int \alpha^a\star G_a^{(0)},\int\beta^b\star\tilde{G}_b\right\}\\&=\int\Big[\frac{c_1}{2}\left(\varepsilon_{bc}[\beta^b,e_1^c]_+\star[\phi,\varepsilon^d{}_a[\alpha^a,\phi_d]_+]_+\right.\\&\quad\left.-[\beta_b,e_1^b]\star[\phi,[\alpha^a,\phi_a]]_+\right)\\&\quad+\frac{c_7}{2}\left(-\varepsilon_{bc}\beta^b\star\varepsilon^d{}_a([\alpha^a,\phi_d]_+\star e_1^c\star\phi+\phi\star e_1^c\star[\alpha^a,\phi_d]_+)\right.\\&\quad\left.+\ \beta_b(\phi\star e_1^b\star[\alpha^a,\phi_a]-[\alpha^a,\phi_a]\star e_1^b\star\phi)\right)\Big]\\&\quad+\text{terms without } e_1^b \text{ or } \phi\,,\end{aligned}\tag{40}$$

The arguments presented above show that if the bracket (39) closes on existing constraints, these constraints are G_3 and G_4, and the structure functions depend on ϕ and ψ. In both G_3 and G_4 the fields e_1^a and ϕ_b appear in the combinations $[\phi_a, e_1^b]$ or $[\phi_a, e_1^b]_+$, i.e. they stay next to each other. Therefore, all terms where ϕ_b and e_1^a appear separated by other fields should vanish. Let us check whether this can be achieved by adjusting the remaining parameters c_1 and c_7. Let us study the terms with ϕ, ϕ_0, α^0, β^0, e_1^0 where α^0 and β^0 stay next to each other, but ϕ_0 and e_1^0 are separated. All such terms in (39) can be easily collected with the help of (40). They read

$$\int\frac{c_1}{2}[\alpha^0,\beta^0]\star(\phi_0\star\phi\star e_1^0-e_1^0\star\phi\star\phi_0)\,.\tag{41}$$

Since they are not allowed we conclude

$$c_1=0\,.\tag{42}$$

Let us now collect all other terms with the same field components where again ϕ_0 and e_1^0 are separated but without any restrictions on the placement of α^0 and β^0:

$$\int\frac{c_7}{2}[e_1^0,\phi]\star(\beta^0\star\phi_0\star\alpha^0-\alpha^0\star\phi_0\star\beta^0)\,.\tag{43}$$

Such terms are also not allowed. Therefore,

$$c_7=0\,.\tag{44}$$

We have just demonstrated that no consistent quadratic deformation of the NCJT model exists. This means that the NCJT model is stable against such deformations.

4 Conclusions

In this paper we studied whether it is possible to deform the action of the NCJT model by adding quadratic terms to the dilaton potential while preserving the number of first-class constraints. The answer we obtained is negative. This, of course, does not exclude the existence of interesting NC gravity models. There is still the possibility of existing other interacting NC dilaton gravities with usual (non-twisted) gauge symmetries. However, it is clear that most of the commutative dilaton gravity models (which admit arbitrary dilaton potentials) cannot be extended to the non-commutative set-up in this approach. Therefore, our results may be considered as a strong argument in favour of the "twisted" approach [9], which allows for practically arbitrary self-interactions of scalar fields. We also point out some earlier results [22] which show that deformations of 2D gravities are trivial if one does not introduce a certain amount of the quantum group structure. Another important result is the construction of twisted conformal symmetries in two dimensions [23]. To incorporate twisted symmetries in the canonical formalism one should probably include twists into the canonical formalism itself.

Finally, since the spherical reduction of higher-dimensional Einstein gravities produces some dilaton gravities in two dimensions, one can expect that our no-go result can be somehow extended to higher dimensions.

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Appendix : Notation and useful identities

Our sign conventions are taken from [1]. We use the tensor $\eta^{ab}=\eta_{ab}=\text{diag}(+1,-1)$ to move indices up and down. The Levi–Civita tensor is defined by $\varepsilon^{01}=-1$, so that the following relations hold

$$\varepsilon^{10}=\varepsilon_{01}=1,\qquad \varepsilon^0{}_1=\varepsilon^1{}_0=-\varepsilon_0{}^1=-\varepsilon_1{}^0=1\,.\tag{A.1}$$

These relations are valid for both ε^{ab} and $\varepsilon^{\mu\nu}$. Note that $\varepsilon^{\mu\nu}$ is always used with both indices up.

The following useful identities hold for arbitrary functions A_1, A_2, B_1 and B_2:

$$\int([A_1,B_1]\star[B_2,A_2]-[B_1,A_2]\star[A_1,B_2])=-\int[A_1,A_2]\star[B_1,B_2]\,,\tag{A.2}$$

$$\int([A_1,B_1]_+\star[A_2,B_2]_+-[A_1,B_2]_+\star[A_2,B_1]_+)=-\int[A_1,A_2]\star[B_1,B_2]\,,\tag{A.3}$$

$$\int([A_1,B_1]_+\star[B_2,A_2]-[B_1,A_2]_+\star[A_1,B_2])=\int[B_1,B_2]\star[A_1,A_2]_+\,,\tag{A.4}$$

$$\int([A_1,B_1]\star[A_2,B_2]-[A_1,B_2]_+\star[A_2,B_1]_+) = -\int[A_1,A_2]_+\star[B_1,B_2]_+ \,. \qquad (A.5)$$

By means of the formula

$$\varepsilon_{ab}\varepsilon_{cd}=\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd} \qquad (A.6)$$

one can get rid of repeated ε-symbols.

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